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On the Limit Lognormal and Other Limit Log-Infinitely Divisible Laws

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Abstract The limit log-infinitely divisible multifractals of Muzy and Bacry (Phys. Rev. E **66**:056121, 2002) are reviewed and shown to possess novel invariance relations that translate into functional Feynman-Kac equations for the corresponding probability distributions. In the special case of the limit lognormal process of Mandelbrot (in *Statistical Models and Turbulence*, M. Rosenblatt, C. Van Atta (Eds.), Springer, New York, 1972), the limit distribution is represented exactly in an operator form using the technique of intermittency expansions. A novel representation for the Mellin transform of the limit distribution is derived and related to the Hurwitz zeta function. For application, the cumulants of the logarithm of the limit lognormal distribution are computed explicitly.

Keywords Multifractals · Intermittency · Infinite divisibility · Selberg integral · Hurwitz zeta · Riemann zeta

1 Introduction

The limit lognormal process was originally introduced and reviewed by Mandelbrot [15, 17] in the context of energy dissipation in intermittent turbulence, and formalized in a series of papers by Kahane [12, 13], [14]. It was re-introduced by Bacry et al. [1], who also constructed in [19] a whole new family of limit log-infinitely divisible multifractal stochastic processes that includes the limit lognormal process as a special case. The interest in this family of processes derives from their remarkable property of stochastic self-similarity with log-infinitely divisible (logID for short) multipliers. In addition, they are grid-free and have stationary increments unlike the canonical multiplicative cascades [16]. Stochastic self-similarity appears in many areas of science under the guise of empirically observed long-range dependence and multiscaling. Examples range from the physics of turbulence [18, 26] to geophysics [25] to human heartbeat dynamics and physiology [10, 11]. Multifractals are an essential mathematical tool for modeling such phenomena. The limit logID processes are

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of additional interest due to their connection with the KPZ formula, confer [5] and [24], and the limit lognormal process specifically as its positive integral moments are given, confer [2], by the celebrated Selberg integral, which generates substantial interest, confer [9].

This paper deals with the outstanding problem of characterizing the distribution of the limit logID processes. Before we describe the contribution of this paper, we summarize briefly what is already known. The limit lognormal process, along with the other limit logID processes, is defined as the zero-scale limit of the exponential functional of the underlying stationary normal (ID in the general case) process with strongly dependent increments. In our previous work [21] we introduced the technique of functional Feynman-Kac equations, which translates the invariances of this underlying process in the normal case with respect to scale, decorrelation length, and intermittency parameters into the corresponding functional equations for the limit lognormal process. This correspondence implies that the scale parameter invariance is equivalent to the property of stochastic self-similarity of the limit lognormal process. The decorrelation length invariance is responsible for how the limit distribution transforms under a particular change of the probability measure. The intermittency parameter invariance quantifies how the limit distribution behaves as a function of the intermittency parameter and thereby captures the limit distribution. We used this invariance in [22] to derive the general rule of intermittency differentiation. This rule in a functional equation for the derivatives of the expectation of an arbitrary smooth function of the limit lognormal distribution with respect to the intermittency parameter. By formally re-summing the resulting Taylor series, we obtained a power series expansion of any such functional with universal coefficients that are independent of the function. In the special case of the Mellin transform [23] we succeeded in computing the coefficients of the corresponding intermittency expansion exactly and showed that it is the small intermittency asymptotic expansion of a particular integral. We summed it by a moment constant method and thus obtained an explicit closed-form formula for the Mellin transform. We then verified that the resulting formula reproduces the known values of the integral moments of the limit lognormal distribution and is in fact the Mellin transform of a valid positive probability distribution. Hence, we effectively introduced a new probability distribution with the properties that its integral moments at arbitrary intermittency and Mellin transform asymptotic in the limit of small intermittency coincide with the corresponding quantities of the limit lognormal distribution. It is our conjecture that the two are one and the same.

The contribution of this paper is to review the theory of limit logID processes as well as the technique of intermittency expansions in the limit lognormal case, and then extend them in several directions. First, we show that the invariances with respect to scale, decorrelation length, and intermittency parameters are not specific to the limit lognormal process and are in fact shared by all limit logID processes. Moreover, we extend the technique of functional Feynman-Kac equations to the general case by translating the first two invariances into the corresponding functional equations and interpreting them in the same way as in the limit lognormal case. Second, we give an exact solution for the intermittency expansion of an arbitrary transform of the limit lognormal distribution by appropriately generalizing the known solution for the Mellin transform. The solution is again regularized by means of its small intermittency asymptotic, and the resulting operator solution is shown to be consistent with the closed-form formula for the Mellin transform. Third, we derive a new formula for the Mellin transform by representing its logarithm in the form of a contour integral with an infinite sum of Hurwitz zeta values in the integrand. This representation allows us to explicitly compute the cumulants of the logarithm of the limit lognormal distribution, which are of particular interest as the moment problem for the logarithm is determinate, unlike the moment problem for the distribution itself, confer [23].

The plan of the paper is as follows. In Sect. 2 we review the general limit logID construction. In Sect. 3 we state and prove invariances of this construction with respect to scale, decorrelation length, and intermittency parameters in Theorem 3.1 and then explain how they can all be translated into the corresponding functional equations. This is carried out in detail in the case of the first two invariances. In Sect. 4 we review our technique of intermittency expansions for the limit lognormal distribution and describe the particular case of the Mellin transform. In Sect. 5 we solve the intermittency expansion of the general transform of the limit lognormal distribution in Corollary 5.1 and provide its regularization, which is consistent with that of the Mellin transform, in Theorem 6.3 in Sect. 6. Section 7 is devoted to the derivation of a new representation of the Mellin transform in Theorem 7.2 and calculation of the cumulants of the logarithm of the limit lognormal distribution in Corollary 7.1. Concluding remarks are presented in Sect. 8.

2 The Limit LogID Construction

We begin this section by giving a review of the limit log-infinitely divisible (logID) construction following Bacry and Muzy [3] and [19], except for several notation-related changes to be explained below. Detailed reviews of the special case of the limit lognormal process can be found in [20] and [21].

The starting point is an ID independently scattered random measure P on the time-scale plane $\mathbb{H}_+ = \{(t, l), l > 0\}$, distributed uniformly with respect to the intensity measure ρ (denoted by μ in [19])

$$\rho(dt\, dl) = dt\, dl/l^2. \tag{1}$$

The infinite divisibility of P means that $P(A)$ is an infinitely divisible random variable for measurable subsets $A \subset \mathbb{H}_+$. The property of being independently scattered means that $P(A)$ and $P(B)$ are independent if A and B do not intersect. Uniform distribution with respect to ρ means that the characteristic function of $P(A)$ is given by

$$\mathbf{E}[e^{iqP(A)}] = e^{\mu\phi(q)\rho(A)}, \quad q \in \mathbb{R}, \tag{2}$$

where $\mu > 0$ is the intermittency parameter[1] and $\phi(q)$ is the logarithm of the characteristic function of the underlying ID distribution and is given by the Lévy-Khinchine formula [6]

$$\phi(q) = imq - \frac{1}{2}q^2 + \int_{\mathbb{R}\setminus\{0\}} [e^{iqy} - 1 - iqy1_{\{|y|<1\}}]\Pi(dy), \tag{3}$$

where the spectral measure $\Pi(y)$ satisfies $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge y^2)\Pi(dy) < \infty$. It is assumed that $\phi(q)$ is extendible to $\Im(q) \geq -1$, which restricts the class of permissible spectral measures. The mean m needs to be chosen in such a way that

$$\phi(-i) = 0 \quad \text{so that } \mathbf{E}[e^{P(A)}] = 1\ \forall A \subset \mathbb{H}_+. \tag{4}$$

For example, in the limit lognormal case, we have

$$\phi(q) = -i\frac{q}{2} - \frac{q^2}{2}. \tag{5}$$

[1] What we call μ is denoted λ^2 in [19]. Also, in [19] it is taken to be part of $\phi(q)$, whereas we find it essential to separate the two. The rationale for this change will become apparent in Sect. 3.

Next, following Schmitt and Marsan [27] and Barral and Mandelbrot [4], Bacry and Muzy [19] introduce special conical sets $\mathcal{A}_{L,\varepsilon}(u)$ in the time-scale plane defined by

$$\mathcal{A}_{L,\varepsilon}(u) = \left\{(t,l) \mid |t-u| \le \frac{l}{2} \text{ for } \varepsilon \le l \le L \text{ and } |t-u| \le \frac{L}{2} \text{ for } l \ge L\right\}. \tag{6}$$

The constant L is the decorrelation length as the sets $\mathcal{A}_{L,\varepsilon}(u)$ and $\mathcal{A}_{L,\varepsilon}(v)$ intersect iff $|u-v| < L$. The last preparatory step is to define a family of ID processes with dependent increments $\omega_{\mu,L,\varepsilon}(u)$ by

$$\omega_{\mu,L,\varepsilon}(u) = P(\mathcal{A}_{L,\varepsilon}(u)). \tag{7}$$

It is clear that $\omega_{\mu,L,\varepsilon}(u)$ and $\omega_{\mu,L,\varepsilon}(v)$ are dependent in general if $|u-v| < L$ and are independent otherwise. With probability one, the process $u \to \omega_{\mu,L,\varepsilon}(u)$ has right-continuous trajectories with finite left limits.

Given these preliminaries, the limit logID process $M_{\mu,L}(t)$ is defined to be the zero scale limit $\varepsilon \to 0$ of finite scale processes $M_{\mu,L,\varepsilon}(t)$ that are themselves defined to be the exponential functional of the process $u \to \omega_{\mu,L,\varepsilon}(u)$

$$M_{\mu,L,\varepsilon}(t) = \int_0^t \exp(\omega_{\mu,L,\varepsilon}(u))\, du. \tag{8}$$

Strictly speaking, $M_{\mu,L,\varepsilon}(dt)$ is a random measure on the real line, whose weak a.s. convergence to a nondegenerate limit measure $M_{\mu,L}(dt)$ was formally established in [3] based on the theory of convergence of a certain class of positive martingales developed by Kahane [13]. Indeed, the martingale property of $\varepsilon \to M_{\mu,L,\varepsilon}(dt)$, namely,

$$\mathbf{E}[M_{\mu,L,\varepsilon'}(t) \mid \mathcal{F}_\varepsilon] = M_{\mu,L,\varepsilon}(t), \quad \varepsilon' < \varepsilon, \tag{9}$$

where $\mathcal{F}_\varepsilon$ is the sigma algebra generated by $P(dt\,dl), l > \varepsilon$, is a direct corollary of the random measure P being independently scattered and (4). The limit measure is nondegenerate in the sense of $\mathbf{E}[M_{\mu,L}(t)] = t$ under the assumption[2] that

$$1 + i\mu\phi'(-i) > 0. \tag{10}$$

The positive moments of $M_{\mu,L}(t)$ are finite under the following necessary and sufficient conditions

$$\begin{aligned} q - \mu\phi(-iq) > 1 \quad &\Longrightarrow \quad \mathbf{E}\big[M^q_{\mu,L}(t)\big] < \infty, \\ \mathbf{E}\big[M^q_{\mu,L}(t)\big] < \infty \quad &\Longrightarrow \quad q - \mu\phi(-iq) \ge 1. \end{aligned} \tag{11}$$

The combination $q - \phi(-iq)$ that appears in (11) is known as the *multiscaling spectrum*. Its significance will be clarified in the next section, confer (32) below. We refer the reader to [3] for further details of their construction and all the proofs.

We conclude our review of the limit logID construction with a fundamental lemma, whose proof the reader can also find in [3]. Let the function $\rho_{L,\varepsilon}(u,v)$ be defined by

$$\rho_{L,\varepsilon}(u,v) = \rho(\mathcal{A}_{L,\varepsilon}(u) \cap \mathcal{A}_{L,\varepsilon}(v)). \tag{12}$$

[2] The condition given in [3] is less stringent than (10), which is however sufficient in most cases of interest such as those of the limit lognormal, compound Poisson, etc. processes.

Clearly, $\rho_{L,\varepsilon}(u,v)$ is an even function of $u-v$ so that we can write $\rho_{L,\varepsilon}(u,v) = \rho_{L,\varepsilon}(|u-v|)$. It is easy to show from (1) and (6) that it is given by

$$\rho_{L,\varepsilon}(u) = \begin{cases} \log(L/|u|) & \text{if } \varepsilon \le |u| \le L, \\ 1+\log(L/\varepsilon) - |u|/\varepsilon & \text{if } |u| < \varepsilon, \end{cases} \tag{13}$$

and it is identically zero for $|u| > L$.

Lemma 2.1 *Given $t_1 \le \cdots \le t_n$ and $q_1, \ldots, q_n$, the joint characteristic function of $\omega_{\mu,L,\varepsilon}(t_i)$, $i = 1, \ldots, n$, is*

$$\mathbf{E}\left[\exp\left(i\sum_{i=1}^{n} q_i \omega_{\mu,L,\varepsilon}(t_i)\right)\right] = \exp\left(\mu \sum_{j=1}^{n}\sum_{k=1}^{j} \alpha_{j,k}\rho_{L,\varepsilon}(t_k - t_j)\right), \tag{14}$$

for some coefficients α_{jk} that involve values of $\phi(q)$ only but not μ, L, ε or t_i. In addition,

$$\sum_{j=1}^{n}\sum_{k=1}^{j} \alpha_{j,k} = \phi\left(\sum_{i=1}^{n} q_i\right). \tag{15}$$

An explicit formula for α_{jk} is given in [3]. The significance of this lemma cannot be overemphasized as it is the source of all known invariances of the $\omega_{\mu,L,\varepsilon}(t)$ process as explained in the next section. In addition, it is easy to see that Lemma 2.1 determines the positive integral moments of $M_{\mu,L}(t)$.

3 Invariances as Functional Feynman-Kac Equations

In this section we consider the link between invariances of the $\omega_{\mu,L,\varepsilon}(t)$ process and functional Feynman-Kac equations, which are known to hold in the limit lognormal case, confer [21], and explain how they can be extended to the general limit logID case.

Introduce a Lévy process (a stochastic process with stationary, independent increments) $\delta \to X(\delta)$ that is independent of the $t \to \omega_{\mu,L,\varepsilon}(t)$ process and defined in terms of the ID distribution associated with $\phi(q)$ as follows

$$\mathbf{E}[e^{iqX(\delta)}] = e^{\delta\phi(q)}, \quad X(0) = 0. \tag{16}$$

The existence and uniqueness of $X(\delta)$ follow from the general theory of ID processes, confer [6]. We now proceed to our result on the general invariances. As in Sect. 2, we write $\omega_{\mu,L,\varepsilon}(t)$ for the process defined in (7) and $\bar{\omega}_{\delta,eL,\varepsilon}(t)$ for an independent copy of this process with the intermittency δ, decorrelation length eL, and scale ε, where e is the base of the natural logarithm.

Theorem 3.1 *Fix μ, L, ε, and δ. Then, there hold the following identities, which are understood to be equalities in law of stochastic processes in t on the interval $t \in [0, L]$,*

$$X(\delta) + \omega_{\mu,L,\varepsilon}(t) = \omega_{\mu,Le^{\delta/\mu},\varepsilon}(t), \tag{17}$$

$$X(\delta) + \omega_{\mu,L,\varepsilon}(t) = \omega_{\mu,L,\varepsilon e^{-\delta/\mu}}(te^{-\delta/\mu}), \tag{18}$$

$$X(\delta) + \omega_{\mu,L,\varepsilon}(t) = \omega_{\mu-\delta,L,\varepsilon}(t) + \bar{\omega}_{\delta,eL,\varepsilon}(t). \tag{19}$$

In (19), δ needs to satisfy $\delta < \mu$.

Their natural interpretation is that (17) is the invariance with respect to the decorrelation length, (18) is the invariance with respect to the scale parameter, and (19) is the invariance with respect to the intermittency parameter. We note that (18) first appeared in an equivalent form in [19]. The other invariances are new.

Proof The proof is based on Lemma 2.1 and the properties of the function $\rho_{L,\varepsilon}(t)$ in (12). Given $|t|<L$, it is easy to see from (13) that $\rho_{L,\varepsilon}(t)$ satisfies

$$\delta+\mu\rho_{L,\varepsilon}(t)=\mu\rho_{Le^{\delta/\mu},\varepsilon}(t), \tag{20}$$

$$\delta+\mu\rho_{L,\varepsilon}(t)=\mu\rho_{L,\varepsilon e^{-\delta/\mu}}(te^{-\delta/\mu}), \tag{21}$$

$$\delta+\mu\rho_{L,\varepsilon}(t)=(\mu-\rho)\rho_{L,\varepsilon}(t)+\delta\rho_{eL,\varepsilon}(t). \tag{22}$$

Each identity translates into the corresponding identity for the $\omega_{\mu,L,\varepsilon}(t)$ process by virtue of Lemma 2.1. For example, to prove (17), pick $t_1<\cdots<t_n<L$ and write

$$\begin{aligned}\mathbf{E}\left[\exp\left(i\sum_{i=1}^{n}q_i\omega_{\mu,Le^{\delta/\mu},\varepsilon}(t_i)\right)\right]&=\exp\left(\mu\sum_{j=1}^{n}\sum_{k=1}^{j}\alpha_{j,k}\rho_{Le^{\delta/\mu},\varepsilon}(t_k-t_j)\right)\\&=\exp\left(\mu\sum_{j=1}^{n}\sum_{k=1}^{j}\alpha_{j,k}\rho_{L,\varepsilon}(t_k-t_j)\right)\exp\left(\delta\sum_{j=1}^{n}\sum_{k=1}^{j}\alpha_{j,k}\right)\\&=\mathbf{E}\left[\exp\left(i\sum_{i=1}^{n}q_i\omega_{\mu,L,\varepsilon}(t_i)\right)\right]\mathbf{E}\left[\exp\left(i\left(\sum_{i=1}^{n}q_i\right)X(\delta)\right)\right].\end{aligned} \tag{23}$$

Thus, the processes on the left- and right-hand sides of (17) have the same finite-dimensional laws on the interval $t\in[0,L]$. Hence they are equal in law because right-continuous processes with finite left limits are completely characterized by their finite-dimensional distributions, confer Sect. 28.4 in [8]. The proof of (18) and (19) goes through verbatim. □

The significance of Theorem 3.1 is that each of the invariances in Theorem 3.1 can be translated into a functional Feynman-Kac equation for the limit process. In other words, we wish to establish some equivalents of the classical Feynman-Kac formula for diffusions in our case of the exponential functional of a strongly nonmarkovian process. The idea of such equations was introduced in [21] in the special case of the underlying ID distribution being normal. Our goal here is to show how such equations can be derived in general.

The starting point is the notion of the generator $\mathcal{L}$ of the Lévy process $X(\delta)$ that is defined by its action on test functions as follows

$$(\mathcal{L}f)(x)=\frac{d}{d\delta}\bigg|_{\delta=0}\mathbf{E}[f(X(\delta)+x)]. \tag{24}$$

An explicit formula for it in terms of the Lévy-Khinchine formula in (3) is well-known, confer [6], Sect. I.2. We only need its action on functions of the special form $f(x)=v(ze^x)$ at $x=0$. It then becomes an integro-differential operator acting on z

$$\begin{aligned}(\mathcal{L}v)(z)&=mzv'(z)+\frac{1}{2}[zv'(z)+z^2v''(z)]\\&\quad+\int_{\mathbb{R}\setminus\{0\}}[v(ze^y)-v(z)-yzv'(z)1_{\{|y|<1\}}]\Pi(dy).\end{aligned} \tag{25}$$

The following proposition translates (17) into the corresponding functional equation. For concreteness, we work with the Laplace transform of $M_{\mu,L}(t)$, that is, given $z>0$ and $t<L$, and dropping μ from the list of arguments for brevity, we set

$$v(z,t,L) = \mathbf{E}[\exp(-zM_{\mu,L}(t))]. \tag{26}$$

From now on it is understood that $\mathcal{L}$ acts only on the z variable as in (25).

Proposition 3.1 *The Laplace transform $v(z,t,L)$ satisfies*

$$(\mathcal{L}v)(z,t,L) = \frac{L}{\mu}\frac{\partial}{\partial L}v(z,t,L). \tag{27}$$

This is the functional form of the invariance in (17).

Proof The idea of the proof is to evaluate the action of the generator $\mathcal{L}$ on the finite-scale Laplace transform $v_\varepsilon(z,t,L) = \mathbf{E}[\exp(-zM_{\mu,L,\varepsilon}(t))]$ in two ways by means of (17) and (24) on the one hand and (25) on the other, and then take the zero scale limit. The action of $\mathcal{L}$ in terms of (25) is immediate

$$\frac{d}{d\delta}\bigg|_{\delta=0}\mathbf{E}[v_\varepsilon(ze^{X(\delta)},t,L)] = (\mathcal{L}v_\varepsilon)(z,t,L). \tag{28}$$

On the other hand, we have by (17)

$$\begin{aligned}\frac{d}{d\delta}\bigg|_{\delta=0}\mathbf{E}[v_\varepsilon(ze^{X(\delta)},t,L)] &= \frac{d}{d\delta}\bigg|_{\delta=0}\mathbf{E}\bigg[\exp\bigg(-ze^{X(\delta)}\int_0^t \exp(\omega_{\mu,L,\varepsilon}(s))\,ds\bigg)\bigg]\\ &= \frac{d}{d\delta}\bigg|_{\delta=0}\mathbf{E}\bigg[\exp\bigg(-z\int_0^t \exp(\omega_{\mu,Le^{\delta/\mu},\varepsilon}(s))\,ds\bigg)\bigg]\\ &= \frac{L}{\mu}\frac{\partial}{\partial L}v_\varepsilon(z,t,L). \end{aligned}\tag{29}$$

Letting $\varepsilon\to 0$ completes the proof. □

The functional form of the invariance in (18) is obtained by the same type of argument starting with (28) and using $\mathbf{E}[v_\varepsilon(ze^{X(\delta)},t,L)] = v_{\varepsilon e^{-\delta/\mu}}(ze^{\delta/\mu},te^{-\delta/\mu},L)$.

Proposition 3.2 *The Laplace transform $v(z,t,L)$ satisfies*

$$(\mathcal{L}v)(z,t,L) = \bigg(\frac{z}{\mu}\frac{\partial}{\partial z} - \frac{t}{\mu}\frac{\partial}{\partial t}\bigg)v(z,t,L). \tag{30}$$

We note that Proposition 3.2 is equivalent to the statement of stochastic self-similarity of the limit process

$$M_{\mu,L}(t) = \frac{t}{L}\exp(X_{\mu\log L/t})M_{\mu,L}(L), \tag{31}$$

understood as the equality of random variables in law at fixed $t<L$. This is the form, in which it was given originally in [19]. It now follows from (16) and (31) that the moments

obey the multiscaling law for q such that $\mathbf{E}[M^q_{\mu,L}(L)] < \infty$

$$\mathbf{E}\big[M^q_{\mu,L}(t)\big] = \left(\frac{t}{L}\right)^{q-\phi(-iq)} \mathbf{E}\big[M^q_{\mu,L}(L)\big], \quad t < L. \tag{32}$$

Hence, $q \to q - \phi(-iq)$ is the multiscaling spectrum of the limit process. It must be emphasized that self-similarity alone says nothing about the joint distribution and does not capture the law of $M_{\mu,L}(L)$ but only of $M_{\mu,L}(t)$ in terms of $M_{\mu,L}(L)$. This is clear from (31). This is also clear from (30) as it can only be solved backward in time.

Proposition 3.1 captures two features of the limit distribution $M_{\mu,L}(L)$ that go beyond self-similarity. These features are most naturally stated in terms of the Laplace transform $v(z, L)$ of $M_{\mu,L}(L)$,

$$v(z, L) = \mathbf{E}[\exp(-zM_{\mu,L}(L))], \tag{33}$$

that is, $v(z, L) = v(z, L, L)$. Also, let us define a new probability measure that is defined in terms of the original probability measure by means of

$$\widetilde{\mathbf{E}}[\cdot] = \lim_{\varepsilon\to 0} \mathbf{E}[\cdot \exp(\omega_{\mu,1,\varepsilon}(1))]. \tag{34}$$

It is well-defined due to (4) and (7).

Corollary 3.1 *The Laplace transform $v(z, L) = v(zL)$ is a function of zL. It satisfies*

$$(\mathcal{L}v)(z) - \frac{z}{\mu}\frac{\partial}{\partial z}v(z) = \frac{z}{\mu}\widetilde{\mathbf{E}}[\exp(-zM_{\mu,1}(1))]. \tag{35}$$

Proof As the left-hand sides of (27) and (30) are the same, we obtain

$$z\frac{\partial}{\partial z}v(z, t, L) - t\frac{\partial}{\partial t}v(z, t, L) = L\frac{\partial}{\partial L}v(z, t, L). \tag{36}$$

On the other hand, it is clear that

$$\frac{\partial}{\partial L}v(z, L) = \frac{d}{dL}v(z, L, L) = \frac{\partial}{\partial t}v(z, t, L)|_{t=L} + \frac{\partial}{\partial L}v(z, t, L)|_{t=L}. \tag{37}$$

It follows from (36) that

$$z\frac{\partial}{\partial z}v(z, L) = L\frac{\partial}{\partial L}v(z, L). \tag{38}$$

Hence, $v(z, L) = v(zL)$. On the other hand, combining (27), (37), and (38), we get

$$\begin{aligned}(\mathcal{L}v)(zL) - \frac{z}{\mu}\frac{\partial}{\partial z}v(zL) &= -\frac{L}{\mu}\lim_{\varepsilon\to 0}\frac{\partial}{\partial t}\bigg|_{t=L}\mathbf{E}\bigg[\exp\bigg(-z\int_0^t e^{\omega_{\mu,L,\varepsilon}(s)}\,ds\bigg)\bigg] \\ &= \frac{zL}{\mu}\lim_{\varepsilon\to 0}\bigg[\exp\bigg(-z\int_0^L e^{\omega_{\mu,L,\varepsilon}(s)}\,ds\bigg)e^{\omega_{\mu,L,\varepsilon}(L)}\bigg]. \end{aligned} \tag{39}$$

The result follows by setting $L = 1$. □

We conclude that the informational content of Proposition 3.1 is that it quantifies how the limit distribution $M_{\mu,1}(1)$ behaves under the change of measure specified by (34). In addition, the fact that $v(z, L)$ is a function of zL means that we have the following equality of random variables in law

$$M_{\mu,L}(L) = LM_{\mu,1}(1). \tag{40}$$

In other words, the dependence of $M_{\mu,L}(L)$ on the decorrelation length is essentially trivial so that (35) involves only the z variable. Finally, we know the distribution of $M_{\mu,L}(t)$ for $t < L$ in terms of that of $M_{\mu,L}(L)$ by Proposition 3.2, hence the primary unknown is $M_{\mu,1}(1)$. Henceforth, we will denote this random variable by M_μ and will write $M_\mu(t)$ for $M_{\mu,1}(t)$.

It is worth making a comment about the appellation 'functional Feynman-Kac.' The 'Feynman-Kac' part comes from the fact that the technique that was illustrated in Propositions 3.1 and 3.2 gives the classical Feynman-Kac equation for the exponential functional of Brownian motion when applied to the invariance $B(s) + \bar{B}(\delta) = B(s + \delta)$ of Brownian motion. The 'functional' part has to do with the fact that (35) is nonlocal, i.e. involves the whole path of the process as opposed to only its value at $t = 1$. This becomes clear if we try to re-write the $\widetilde{\mathbf{E}}$ expectation with respect to the original measure. We refer the reader to [21, Eq. (47)], where this is done explicitly in the limit lognormal case. Nonlocality is also apparent in the next section.

4 Review of Intermittency Differentiation

The distribution of M_μ is determined by the intermittency parameter invariance in (19). Conceptually, it is easy to see that by evaluating the derivative in (28) in two ways as we did in the proof of Proposition 3.1 except now using (19) instead of (17), we obtain an equation for intermittency differentiation. By iterating this equation, we then obtain a formal power series expansion with some universal coefficients depending only on the underlying ID distribution. These coefficients can subsequently be determined by applying the expansion to the positive integral moments of M_μ, which can be computed using Lemma 2.1. This procedure thus recovers the limit distribution from the dependence of its moments on the intermittency parameter, at least formally. The resulting formal intermittency expansion may or may not be convergent and needs to be regularized in the latter case, which is the main drawback of our method. In this section we will summarize how this method works in the limit lognormal case and develop our operator solution in Sect. 5. Regularization will be treated in Sect. 6.

The positive integral moments of M_μ were shown in [2] to be given by the celebrated Selberg integral, confer [28]. Given integral l such that $2 \le l < 2/\mu$,

$$\mathbf{E}[M_\mu^l] = \int_0^1 \cdots \int_0^1 \prod_{i<j}^{l} |s_i - s_j|^{-\mu} \, d\mathbf{s}^{(l)} = \prod_{k=0}^{l-1} \frac{\Gamma(1-(k+1)\mu/2)\Gamma^2(1-k\mu/2)}{\Gamma(1-\mu/2)\Gamma(2-(l+k-1)\mu/2)}, \tag{41}$$

which from now on we will denote by $S_l(\mu)$.

Let $F(x)$ be an arbitrary smooth function that does not involve the intermittency parameter $0 \le \mu < 1$[3] and let $F^{(k)}(x)$ denote its kth derivative. Our results on general intermittency expansions established [22] are summarized in the following propositions.

[3]Nondegeneracy is guaranteed by $\mu < 2$, confer (10), the restriction of $\mu < 1$ ensures the finiteness of the 2nd moment, confer (11).

Consider the expectation of a general functional of the limit lognormal process

$$v(\mu, f, F) \triangleq \mathbf{E}\left[F\left(\int_0^1 e^{\mu f(s)}\, dM_\mu(s)\right)\right], \tag{42}$$

where $f(s)$ is an arbitrary continuous function that does not involve μ. This functional is path-dependent unless $f \equiv 0$, its somewhat peculiar functional form is motivated by the fact that this functional form is invariant under intermittency differentiation. The integration with respect to the limit measure $dM_\mu(s)$ is understood in the sense of $\varepsilon \to 0$ limit so that $v(\mu, f, F) = \lim_{\varepsilon\to 0} v_\varepsilon(\mu, f, F)$ and $v_\varepsilon(\mu, f, F) \triangleq \mathbf{E}[F(\int_0^1 e^{\mu f(s)} dM_{\mu,\varepsilon}(s))]$ with $dM_{\mu,\varepsilon}(s)$ as in (8) with $L = 1$. Also, let $g(s_1, s_2)$ be defined by

$$g(s_1, s_2) \triangleq -\log|s_1 - s_2|. \tag{43}$$

Its significance is that $\lim_{\varepsilon\to 0} \mathbf{Cov}(\omega_{\mu,1,\varepsilon}(s_1), \omega_{\mu,1,\varepsilon}(s_2)) = \mu\, g(s_1, s_2)$ on $0 < |s_1 - s_2| < 1$. Finally, we will use $\otimes k$ to denote the k-dimensional unit cube $[0, 1] \times \cdots \times [0, 1]$. Then, we have the following general *rule of intermittency differentiation*.

Theorem 4.1 *The expectation $v(\mu, f, F)$ is invariant under intermittency differentiation and satisfies*

$$\begin{aligned} \frac{\partial}{\partial\mu} v(\mu, f, F) = &\int_{\otimes 1} v(\mu, f + g(\cdot, s), F^{(1)}) e^{\mu f(s)} f(s)\, ds \\ &+ \frac{1}{2}\int_{\otimes 2} v(\mu, f + g(\cdot, s_1) + g(\cdot, s_2), F^{(2)}) \\ &\times e^{\mu(f(s_1)+f(s_2)+g(s_1,s_2))} g(s_1, s_2)\, d\mathbf{s}^{(2)}. \end{aligned} \tag{44}$$

The mathematical content of (44) is that differentiation with respect to the intermittency parameter μ is equivalent to a combination of two functional shifts induced by the g function. The single-integral term corresponds to the exponential prefactor in (42), whereas the double-integral term corresponds to the intrinsic dependence of $M_\mu(t)$ on μ. It is clear that both terms in (44) are of the same functional form as the original functional in (42) so that Theorem 4.1 allows us to compute derivatives of all orders.

Proposition 4.1 *The expectation $\mathbf{E}[F(M_\mu)]$ has the formal expansion*

$$\mathbf{E}[F(M_\mu)] = F(1) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!}\left[\sum_{k=2}^{2n} F^{(k)}(1) H_{n,k}\right]. \tag{45}$$

The universal expansion coefficients $H_{n,k}$, $n = 1, 2, 3, \ldots,$ are given by the binomial transform of the derivatives of the Selberg integral

$$H_{n,k} = \frac{(-1)^k}{k!} \sum_{l=2}^{k} (-1)^l \binom{k}{l} \frac{\partial^n S_l}{\partial\mu^n}\bigg|_{\mu=0}. \tag{46}$$

Proposition 4.2 *The expansion coefficients $H_{n,k}$ satisfy*

$$H_{n,k} = 0 \quad \forall k > 2n. \tag{47}$$

Corollary 4.1 *There holds the following formal expansion in terms of the derivatives of moment-related expectations of the process*

$$\mathbf{E}[F(M_\mu)] = F(1) + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} \left[\sum_{k=2}^{2n} \frac{F^{(k)}(1)}{k!} \frac{\partial^n}{\partial \mu^n}\bigg|_{\mu=0} \mathbf{E}[(M_\mu - 1)^k] \right]. \tag{48}$$

The representation in (48) reveals an essential feature of the structure of our expansions. We see that (48) is an exactly renormalized expansion in the moments of M_μ. Indeed, it is easy to see that if the positive moments of all orders were finite and Taylor expandable in μ, then (48) would be the same as the naive expansion[4] in the moments $\mathbf{E}[F(M_\mu)] = F(1) + \sum_{k=1}^{\infty} F^{(k)}(1)\mathbf{E}[(M_\mu - 1)^k]/k!$. Unlike the naive expansion, however, all the coefficients in (48) are finite because the derivatives are taken at $\mu = 0$, and $S_l(\mu)$ is finite so long as $l < 2/\mu$. Moreover, our renormalization is exact as the expansion in (48) is not ad hoc but is rather derived from the exact functional equation in Theorem 4.1.

The main result of [23] was to explicitly calculate the intermittency expansion for the Mellin transform (complex moments) of M_μ. The moments correspond to $F(x) = x^q$ in (45) for some given $q \in \mathbb{C}$. By Proposition 4.1, the intermittency expansion for the moments is

$$\mathbf{E}\big[M_\mu^q\big] = 1 + \sum_{n=1}^{\infty} \frac{\mu^n}{n!} f_n(q), \tag{49}$$

$$f_n(q) = \sum_{k=2}^{\infty} (q)_k\, H_{n,k}, \quad n = 1, 2, 3, \ldots. \tag{50}$$

As usual, $(q)_k$ denotes the 'falling factorial' $(q)_k \triangleq q(q-1)(q-2)\cdots(q-k+1)$. Note that the upper limit of summation has been extended to infinity by Proposition 4.2. As usual, $\zeta(s)$[5] denotes the Riemann zeta function, $B_n(s)$ the nth Bernoulli polynomial, and $Y_n(x_1, \ldots, x_n)$ the complete exponential Bell polynomial of order n.

Theorem 4.2 *Let $f_0(q) = 1$ and define the polynomials $b_r(q)$, $r = 0, 1, 2, \ldots$*

$$\begin{aligned} b_r(q) = \frac{1}{2^{r+1}} \bigg[\zeta(r+1) \bigg[\frac{B_{r+2}(q+1) + 2B_{r+2}(q) - 3B_{r+2}}{r+2} - q \bigg] \\ + (\zeta(r+1) - 1) \bigg[\frac{B_{r+2}(q-1) - B_{r+2}(2q-1)}{r+2} \bigg] \bigg]. \end{aligned} \tag{51}$$

Then, $f_n(q)$ satisfies the recurrence

$$f_{n+1}(q) = n! \sum_{r=0}^{n} \frac{f_{n-r}(q)}{(n-r)!} b_r(q) \tag{52}$$

and is given explicitly in terms of Y_n by

$$f_n(q) = Y_n(b_0(q)0!, b_1(q)1!, \ldots, b_{n-1}(q)(n-1)!). \tag{53}$$

[4]To see this, note that the k sum in (48) can be extended to infinity due to $\partial^n/\partial\mu^n\,|_{\mu=0}\,\mathbf{E}[(M_\mu - 1)^k] \equiv k!\,H_{n,k} = 0\ \forall k > 2n$ and $n \ge 1$ by Proposition 4.2.

[5]We will write $\zeta(1)$ to denote Euler's constant. It never enters any of the final formulas as the coefficient it multiplies is identically zero throughout this paper.

The moments have the following exact formal representation

$$\mathbf{E}\big[M_\mu^q\big] = \exp\left(\sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r+1}\, b_r(q)\right), \quad q \in \mathbb{C}. \tag{54}$$

The series $\sum_{r=0}^{\infty} \mu^{r+1}\, b_r(q)/(r+1)$ is divergent in general with the exception of a finite range of positive and negative integral q, confer Theorem 6.1 below. This means that the Mellin transform of M_μ is not analytic in the intermittency parameter, and (54) ought to be interpreted as its asymptotic expansion. We will regularize this expansion along with the general expansion in (58) below in Sect. 6.

5 Operator Solution of the General Transform

In this section we will determine the intermittency expansion of the general transform by extending the Mellin transform recurrence in Theorem 4.2 and then solving the new recurrence in an operator form.

Consider the general transform of the form $\mathbf{E}[G(s+\log M_\mu)]$ for some fixed constant s. The corresponding intermittency expansion is

$$\mathbf{E}[G(s+\log M_\mu)] = \sum_{n=0}^{\infty} G_n(s)\frac{\mu^n}{n!}. \tag{55}$$

The main result of this section is the following theorem and its corollary, which completely characterize the expansion in (55). Recall the definition of the polynomials $b_r(q)$ in (51).

Theorem 5.1 *The coefficients $G_n(s)$ of the general intermittency expansion satisfy the recurrence*

$$G_{n+1}(s) = \sum_{r=0}^{n} \frac{n!}{(n-r)!} b_r\left(\frac{d}{ds}\right) G_{n-r}(s), \quad G_0(s) = G(s). \tag{56}$$

Corollary 5.1

$$G_n(s) = Y_n\left(0!b_0\left(\frac{d}{ds}\right), \ldots, (n-1)!b_{n-1}\left(\frac{d}{ds}\right)\right) G(s), \tag{57}$$

$$\mathbf{E}[G(s+\log M_\mu)] = \exp\left(\sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r+1}\, b_r\left(\frac{d}{ds}\right)\right) G(s). \tag{58}$$

The proof will be given in a series of lemmas. We begin with a lemma that gives a closed-form expression for the universal expansion coefficients $H_{n,k}$. Let s_{lk} and S_{lk} denote Stirling numbers of the first and second kind, respectively, the polynomials $f_n(q)$ be as in (53), and $f_n^{(l)}(0)$ denote their derivatives at $q=0$.

Lemma 5.1 *The universal expansion coefficients $H_{n,k}$ satisfy*

$$H_{n,k} = \sum_{l=2}^{2n} \frac{S_{lk}}{l!} f_n^{(l)}(0), \quad k = 2, 3, 4, \ldots. \tag{59}$$

Proof The starting point is (50). Differentiating it with respect to q, we obtain

$$f_n^{(l)}(0) = l! \sum_{k=2}^{\infty} s_{kl}\, H_{n,k}. \tag{60}$$

The upper limit of summation is formally extended to infinity by Proposition 4.2. The result now follows by the inversion property of Stirling numbers. □

Corollary 5.2 *The derivatives $f_n^{(l)}(0)$ satisfy*

$$f_n^{(l)}(0) = 0 \quad \forall l > 2n. \tag{61}$$

This follows from Proposition 4.2, (60), and the fact that $s_{kl} = 0$ if $k < l$.

Lemma 5.2 *The coefficients $G_n(s)$ satisfy*

$$G_n(s) = \sum_{l=0}^{2n} \frac{G^{(l)}(s)}{l!} f_n^{(l)}(0). \tag{62}$$

Proof Fix s and define $F(x) = G(s + \log x)$. Then, the coefficients $F^{(k)}(1)$ that enter (45) are given by

$$F^{(k)}(1) = \sum_{p=0}^{k} s_{kp}\, G^{(p)}(s). \tag{63}$$

Substituting this equation and (59) into the intermittency expansion, extending the upper limit of the l sum to infinity by Corollary 5.2, changing the order of summation, and making another use of Stirling inversion, we obtain

$$\begin{aligned}
G_n(s) &= \sum_{k=2}^{\infty} H_{nk} \left[\sum_{p=0}^{k} s_{kp}\, G^{(p)}(s) \right] \\
&= \sum_{l=2}^{\infty} \frac{1}{l!} f_n^{(l)}(0) \left[\sum_{k=2}^{\infty} \sum_{p=0}^{k} S_{lk}\, s_{kp}\, G^{(p)}(s) \right] \\
&= \sum_{l=2}^{\infty} \frac{1}{l!} f_n^{(l)}(0) [G^{(l)}(s) - G^{(1)}(s)].
\end{aligned} \tag{64}$$

Finally, the polynomials $f_n(q)$ have the property

$$f_n(0) = f_n(1) = 0 \quad \forall n \geq 1. \tag{65}$$

Hence,

$$f_n^{(1)}(0) = -\sum_{l=2}^{\infty} \frac{1}{l!} f_n^{(l)}(0) \quad \forall n \geq 1. \tag{66}$$

The result follows. □

We can now give the proof of Theorem 5.1.

Proof By Lemma 5.2 and Corollary 5.2 we can write

$$G_{n+1}(s)=\sum_{l=0}^{\infty}\frac{G^{(l)}(s)}{l!}\,f_{n+1}^{(l)}(0). \tag{67}$$

Substituting (52) and changing the order of summation (all the sums involved are finite despite notation), we have

$$\begin{aligned}
G_{n+1}(s)&=n!\sum_{r=0}^{n}\frac{1}{(n-r)!}\left[\sum_{p=0}^{\infty}b_r^{(p)}(0)\sum_{l=p}^{\infty}\binom{l}{p}\frac{G^{(l)}(s)}{l!}\,f_{n-r}^{(l-p)}(0)\right]\\
&=n!\sum_{r=0}^{n}\frac{1}{(n-r)!}\left[\sum_{p=0}^{\infty}b_r^{(p)}(0)\frac{1}{p!}\sum_{l=0}^{\infty}\frac{G^{(p+l)}(s)}{l!}\,f_{n-r}^{(l)}(0)\right]\\
&=n!\sum_{r=0}^{n}\frac{1}{(n-r)!}\left[\sum_{p=0}^{\infty}b_r^{(p)}(0)\frac{1}{p!}\frac{d^p}{ds^p}\right]G_{n-r}(s). \qquad (68)
\end{aligned}$$

The last line was obtained by differentiating (62) with the index $n-r$ p times. The result follows. □

The proof of Corollary 5.1 is now quite simple.

Proof The key observation is that the operators $b_r(d/ds)$ are commuting because $b_r(q)$ are polynomials with constant coefficients in q, confer (51). Now, the recursion relation of complete Bell polynomials, confer [7], Chap. 11, is

$$Y_{n+1}(x_1,\ldots,x_{n+1})=\sum_{r=0}^{n}\binom{n}{r}Y_{n-r}(x_1,\ldots,x_{n-r})x_{r+1},\qquad Y_0=1. \tag{69}$$

This recursion shows that (57) follows from (56). Equation (58) follows from the well-known formula for the generating function of Y_n, confer [7], Chap. 11. □

The significance of Theorem 5.1 is that it generalizes the recurrence in Theorem 4.2 and shows that the polynomials $b_r(q)$ play a fundamental role not only for the Mellin transform but in fact for the general transform. The analogy of the general and Mellin transforms extends much further in that the solution for the general transform in Corollary 5.1 is obtained by replacing q with d/ds in the solution for the Mellin transform in (53) and (54). Thus, (58) is an exact *operator solution* for the general intermittency expansion in (55). It is obvious that (58) reproduces the Mellin transform expansion in (54) by taking $G(s)=\exp(qs)$ for some fixed $q\in\mathbb{C}$ and using

$$b_r\left(\frac{d}{ds}\right)e^{qs}=b_r(q)\,e^{qs}. \tag{70}$$

6 Regularization

The solution for the general intermittency expansion in Corollary 5.1 is formal and needs to be regularized, i.e. we need to sum the divergent series in (58). The regularized solution must correspond to a valid positive probability distribution having the correct integral moments given by the Selberg integral in (41). In addition, it must reproduce (58) as its asymptotic expansion in the small intermittency limit. A solution with these properties is given in Theorem 6.3 below. The question of uniqueness of such a solution is open, hence we will write $\widetilde{M}_\mu$ to distinguish our construction from the limit lognormal distribution as defined in Sect. 2.

We begin by regularizing the expansion for the Mellin transform in (54). In [23] we proposed to sum it by means of

$$\sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r+1} b_r(q) \sim \int_0^\infty \Bigg[\frac{1}{e^x-1}\Bigg[\frac{e^{\frac{\mu x}{2}(q+1)} + 2e^{\frac{\mu x}{2}q} - 3 + e^{\frac{\mu x}{2}(q-1)} - e^{\frac{\mu x}{2}(2q-1)}}{e^{\frac{\mu x}{2}}-1} - (1+q+qe^{\frac{\mu x}{2}})\Bigg] + e^{-x}\Bigg[\frac{e^{\frac{\mu x}{2}(2q-1)} - e^{\frac{\mu x}{2}(q-1)}}{e^{\frac{\mu x}{2}}-1} - q\Bigg]\Bigg]\frac{dx}{x} \triangleq \mathcal{D}(q). \tag{71}$$

The meaning of '$\sim$' is that the series $\sum_{r=0}^{\infty} \mu^{r+1} b_r(q)/(r+1)$ is the asymptotic expansion in μ of the integral on the right-hand side of (71) that we denote by $\mathcal{D}(q)$. The integral is convergent for $\Re(q) < 2/\mu$. Hence, we *define* the Mellin transform by

$$\mathbf{E}\big[\widetilde{M}_\mu^q\big] \triangleq \exp(\mathcal{D}(q)), \quad \Re(q) < \frac{2}{\mu}. \tag{72}$$

The motivation for this particular way of summing the divergent series in (54) is explained in the following theorems that were established in [23].

Theorem 6.1 *For real integral values of q such that $-2/\mu + 1/2 < q < 2/\mu$ the series in (54) and convergent and satisfies*

$$\sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r+1} b_r(q) = \mathcal{D}(q). \tag{73}$$

In addition, for positive integral q such that $q < 2/\mu$ we have

$$\exp(\mathcal{D}(q)) = \prod_{k=0}^{l-1} \frac{\Gamma(1-(k+1)\mu/2)\Gamma^2(1-k\mu/2)}{\Gamma(1-\mu/2)\Gamma(2-(l+k-1)\mu/2)}. \tag{74}$$

Theorem 6.2 *Given $0<\mu<1$, the function $q \to \exp(\mathcal{D}(iq))$, $q \in \mathbb{R}$, is the characteristic function of an infinitely divisible distribution.*

In summary, Theorem 6.2 says that for every $0 < \mu < 1$ $\exp(\mathcal{D}(iq))$ is the characteristic function of some random variable that we call $\log \widetilde{M}_\mu$. In other words, $\mu \to \widetilde{M}_\mu$ is a family of random variables that are parameterized by μ. Hence $\exp(\mathcal{D}(q))$ is the Mellin transform of the random variable $\widetilde{M}_\mu$, whose positive integral moments coincide with the known moments of the limit lognormal distribution by Theorem 6.1 for every μ. Finally, the small intermittency expansion of this Mellin transform coincides with the intermittency expansion of the limit lognormal distribution by construction $\mathbf{E}[\widetilde{M}_\mu^q] \sim \exp(\sum_{r=0}^\infty \mu^{r+1} b_r(q)/(r+1))$ as $\mu \to +0$. We conjecture that the two are the same $M_\mu = \widetilde{M}_\mu$.

We can now state the main result of this section, which gives the regularized solution for the general transform in an operator form.

Theorem 6.3 *Let $\widetilde{M}_\mu$ be the probability distribution defined by (72). Then,*

$$\mathbf{E}[G(s+\log\widetilde{M}_\mu)] = \exp\Big(\mathcal{D}\Big(\frac{d}{ds}\Big)\Big)G(s). \tag{75}$$

The small intermittency asymptotic expansion of $\mathbf{E}[G(s+\log\widetilde{M}_\mu)]$ coincides with the intermittency expansion in (58)

$$\mathbf{E}[G(s+\log\widetilde{M}_\mu)] \sim \sum_{n=0}^\infty G_n(s)\frac{\mu^n}{n!} \quad \textit{as } \mu\to+0. \tag{76}$$

The action of the operator $\mathcal{D}(d/ds)$ is given by

$$\begin{aligned}\Big(\mathcal{D}\Big(\frac{d}{ds}\Big)f\Big)(s) = \int_0^\infty \frac{dx}{x}\Big[\frac{1}{e^x-1}\Big[&\frac{e^{\frac{\mu x}{2}}f(s+\mu x/2)+2f(s+\mu x/2)-3f(s)}{e^{\frac{\mu x}{2}}-1} \\ &+\frac{e^{-\frac{\mu x}{2}}f(s+\mu x/2)-e^{-\frac{\mu x}{2}}f(s+2\mu x/2)}{e^{\frac{\mu x}{2}}-1}-\Big(1+\frac{df}{ds}+e^{\frac{\mu x}{2}}\frac{df}{ds}\Big)\Big] \\ &+e^{-x}\Big[\frac{e^{-\frac{\mu x}{2}}f(s+2\mu x/2)-e^{-\frac{\mu x}{2}}f(s+\mu x/2)}{e^{\frac{\mu x}{2}}-1}-\frac{df}{ds}\Big]\Big].\end{aligned} \tag{77}$$

Proof The formula for $\mathcal{D}(d/ds)$ in (77) is immediate from (71). It follows that exponentials are its eigenfunctions

$$\mathcal{D}\Big(\frac{d}{ds}\Big)e^{qs} = \mathcal{D}(q)e^{qs}, \tag{78}$$

for any fixed $q\in\mathbb{C}$. Now, the proof of (75) follows from Fourier inversion. Indeed, by Theorem 6.2, the probability density function of $\log\widetilde{M}_\mu$ is

$$\mathrm{pdf}_{\log\widetilde{M}_\mu}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iqx}e^{\mathcal{D}(iq)}\,dq. \tag{79}$$

Denote $(\mathcal{F}G)(q)$ the Fourier transform of $G(s)$

$$(\mathcal{F}G)(q) \triangleq \int_{\mathbb{R}} e^{-iqs}G(s)\,ds. \tag{80}$$

Then, the left-hand side of (75) is

$$\mathbf{E}[G(s+\log\widetilde{M}_\mu)] = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iqs}e^{\mathcal{D}(iq)}(\mathcal{F}G)(q)\,dq. \tag{81}$$

This is exactly the same as the right-hand side of (75) if we express $G(s)$ as the inverse Fourier transform of $(\mathcal{F}G)(q)$ and recall (78).

The proof of (76) was effectively given in Sect. 5. In fact, as Corollary 5.1 and (71) involve the same infinite series, the result follows from the fact that this series is the small intermittency asymptotic expansion of $\mathcal{D}(q)$. □

The action of the operator $\mathcal{D}(d/ds)$ determines the distribution of $\widetilde{M}_\mu$ uniquely. The actual calculation of its action is as difficult as the task of Fourier inverting $\exp(\mathcal{D}(iq))$. For this reason, it is interesting to characterize the distribution in an alternative way by means of a set of invariants that capture it uniquely. One such set of invariants is the set of all positive integral moments of $\log \widetilde{M}_\mu$, for we showed in [23] that the associated moment problem is determinate. Their computation is presented in the next section.

7 Calculation of the Cumulants

In this section we are interested in the positive integral moments of $\log \widetilde{M}_\mu$. Faà di Bruno's formula and (72) imply that they can be expressed as exponential Bell polynomials

$$\mathbf{E}[(\log \widetilde{M}_\mu)^n] = Y_n(\mathcal{D}^{(1)}(0), \ldots, \mathcal{D}^{(n)}(0)). \tag{82}$$

The quantity $\mathcal{D}^{(p)}(0)$ denotes the pth derivative of $\mathcal{D}(q)$ with respect to q at $q = 0$. The moments can also be computed recursively by means of (69)

$$\mathbf{E}[(\log \widetilde{M}_\mu)^{n+1}] = \sum_{r=0}^{n} \binom{n}{r} \mathbf{E}[(\log \widetilde{M}_\mu)^{n-r}] \mathcal{D}^{(r+1)}(0). \tag{83}$$

Hence, it is sufficient to compute $\mathcal{D}^{(p)}(0)$, i.e. the *cumulants* of $\log \widetilde{M}_\mu$.

It is worth emphasizing that $\mathcal{D}(q)$ is also a function of the intermittency parameter μ. The dependence of $\mathcal{D}(q)$ on μ is not analytic and leads to the asymptotic intermittency expansions that we considered in Sect. 5. On the other hand, $\mathcal{D}(q)$ is analytic in q on $\Re(q) < 2/\mu$. It is this dependence that we will consider in this section.

The starting point of our analysis is the following formula for the Mellin transform that we established in [23], which extends Selberg's finite product to an infinite product of gamma factors.

Theorem 7.1 *Let $\mathcal{D}(q)$ be as in (71) and $\Re(q) < 2/\mu$. Then,*

$$\exp(\mathcal{D}(q)) = \left(\frac{2}{\mu}\right)^q \Gamma(1-\mu q/2)\Gamma^{-q}(1-\mu/2) \frac{\Gamma(2+2/\mu-2q)}{\Gamma(2+2/\mu-q)} \tag{84}$$

$$\times \prod_{n=1}^{\infty} \left(\frac{2n}{\mu}\right)^{2q} \frac{\Gamma^3(1-q+2n/\mu)}{\Gamma^3(1+2n/\mu)} \frac{\Gamma(2-q+2n/\mu)}{\Gamma(2-2q+2n/\mu)}. \tag{85}$$

We are now primarily interested in the logarithm of the infinite product term in (85) because it is the core of the structure of the Mellin transform.

Theorem 7.2 *Let q be such that $|\arg(-q)| < \pi$ and $\Re(q) < 1/2 + 1/\mu$. Then,*

$$\log\left[\prod_{n=1}^{\infty} \left(\frac{2n}{\mu}\right)^{2q} \frac{\Gamma^3(1-q+2n/\mu)}{\Gamma^3(1+2n/\mu)} \frac{\Gamma(2-q+2n/\mu)}{\Gamma(2-2q+2n/\mu)}\right] \tag{86}$$

$$= q\mu\int_0^\infty\left[\frac{e^{-t}}{2t}+\frac{1}{e^t-1}\left[\frac{1}{e^{\mu t/2}-1}-\frac{2}{\mu t}\right]\right]dt+\log\frac{\Gamma(1+\frac{\mu}{2}(1-2q))}{\Gamma(1+\frac{\mu}{2}(1-q))} \tag{87}$$

$$-\frac{2}{\pi i}\int_{5/2-i\infty}^{5/2+i\infty}\frac{\pi(-q)^s}{s\sin(\pi s)}\left(\sum_{k=1}^\infty\zeta(s,1+2k/\mu)\right)(1-2^{s-2})\,ds. \tag{88}$$

Theorem 7.2 says that the logarithm of the infinite product term in (85) equals the sum of a term that is linear in q, a term involving two log-gamma functions, and a contour integral involving an infinite sum of Hurwitz zeta values. This sum $\sum_{k=1}^\infty\zeta(s,1+2k/\mu)$ can thus be interpreted as being the core of the structure of the Mellin transform. It is bounded on the contour so that the integral is convergent.

Corollary 7.1 *The cumulants of* $\log\widetilde{M}_\mu$ *are*

$$\mathcal{D}^{(1)}(0)=\log\left(\frac{2}{\mu}\right)-\log\Gamma\left(1-\frac{\mu}{2}\right)-\psi\left(2+\frac{2}{\mu}\right)-\frac{\mu}{2}\psi\left(1+\frac{\mu}{2}\right)-\frac{\mu}{2}\psi(1)$$

$$+\mu\int_0^\infty\left[\frac{e^{-t}}{2t}+\frac{1}{e^t-1}\left[\frac{1}{e^{\mu t/2}-1}-\frac{2}{\mu t}\right]\right]dt, \tag{89}$$

$$\mathcal{D}^{(p)}(0)=(p-1)!\left[(2^p-1)\zeta\left(p,2+\frac{2}{\mu}\right)+\left(\frac{\mu}{2}\right)^p(2^p-1)\zeta\left(p,1+\frac{\mu}{2}\right)\right.$$

$$\left.+\left(\frac{\mu}{2}\right)^p\zeta(p)+4(1-2^{p-2})\sum_{k=1}^\infty\zeta\left(p,1+\frac{2k}{\mu}\right)\right],\quad p=2,3,4,\ldots. \tag{90}$$

In view of the fact that the moment problem of $\log\widetilde{M}_\mu$ is determinate, the cumulants $\mathcal{D}^{(p)}(0)$ capture the distribution of $\log\widetilde{M}_\mu$ uniquely.

Proof Let $a_k\triangleq 1+2k/\mu$. Then the logarithm of the infinite product in (86) is the limit of the sum S_N

$$\sum_{k=1}^N[2q\log(a_k-1)+3\log\Gamma(a_k-q)/\Gamma(a_k)+\log\Gamma(1+a_k-q)/\Gamma(1+a_k-2q)] \tag{91}$$

as $N\to\infty$. It is easy to show that this sum equals

$$S_N=\sum_{k=1}^N[2q\log(a_k-1)+\log(a_k-q)-\log(a_k-2q)-2q\psi(a_k)$$

$$+4(\log\Gamma(a_k-q)/\Gamma(a_k)+q\psi(a_k))-(\log\Gamma(a_k-2q)/\Gamma(a_k)+2q\psi(a_k))]. \tag{92}$$

We now need the following identity, which is an extension of an identity that appears in Sect. 13.6 of [30],

$$\log\Gamma(a+z)-\log\Gamma(a)=z\psi(a)+\frac{z^2}{2}\zeta(2,a)-\frac{1}{2\pi i}\int_{5/2-i\infty}^{5/2+i\infty}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)\,ds. \tag{93}$$

The original identity involves the contour $\Re(s)=3/2$, which we shifted to $\Re(s)=5/2$ by picking up the residue at $s=2$. In general, we have

$$\operatorname{Res}\left[\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a), s=m\right]=\frac{(-z)^m}{m}\zeta(m,a),\qquad m=2,3,4,\ldots. \tag{94}$$

The identity in (93) holds for $a>0$, $|\arg(z)|<\pi$. It follows that the limit equals

$$\lim_{N\to\infty} S_N=\sum_{k=1}^{\infty}[2q\log(a_k-1)+\log(a_k-q)/(a_k-2q)-2q\psi(a_k)] \tag{95}$$

$$-\frac{2}{\pi i}\int_{5/2-i\infty}^{5/2+i\infty}\frac{\pi(-q)^s}{s\sin(\pi s)}\left(\sum_{k=1}^{\infty}\zeta(s,a_k)\right)(1-2^{s-2})\,ds. \tag{96}$$

Using the well-known identities, confer Chap. 3 of [29],

$$\psi(1+z)=\log(z)+\int_0^\infty e^{-tz}\left(\frac{1}{t}-\frac{1}{e^t-1}\right)dt,\qquad \Re(z)>0, \tag{97}$$

$$\log(z)=\int_0^\infty (e^{-t}-e^{-tz})\,\frac{dt}{t},\qquad \Re(z)>0, \tag{98}$$

we get

$$2q(\log(a_k-1)-\psi(a_k))=2q\int_0^\infty e^{-t(a_k-1)}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt, \tag{99}$$

$$\log(a_k-q)-\log(a_k-2q)=\int_0^\infty \frac{dt}{t}[e^{-t(a_k-2q)}-e^{-t(a_k-q)}]. \tag{100}$$

Summing over k, we obtain for the infinite sum in (95)

$$\int_0^\infty\left[2q\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\frac{1}{e^{2t/\mu}-1}+(e^{-t(1-2q)}-e^{-t(1-q)})\frac{1}{t(e^{2t/\mu}-1)}\right]dt. \tag{101}$$

Changing variables $t'=2t/\mu$, we get

$$\int_0^\infty\left[\mu q\left(\frac{1}{e^{\mu t/2}-1}-\frac{2}{\mu t}\right)\frac{1}{e^t-1}+(e^{-\mu t(1-2q)/2}-e^{-\mu t(1-q)/2})\frac{1}{t(e^t-1)}\right]dt. \tag{102}$$

Finally, by Malmsten's formula,

$$\log\Gamma(1+z)=\int_0^\infty\left(\frac{e^{-tz}-1}{e^t-1}+ze^{-t}\right)\frac{dt}{t},\qquad \Re(z)>-1, \tag{103}$$

$$\log\frac{\Gamma(1+\frac{\mu}{2}(1-2q))}{\Gamma(1+\frac{\mu}{2}(1-q))}=\int_0^\infty\left[-q\frac{\mu}{2}e^{-t}+\frac{e^{-\mu t(1-2q)/2}-e^{-\mu t(1-q)/2}}{e^t-1}\right]\frac{dt}{t}. \tag{104}$$

The result follows.

The proof of Corollary 7.1 is straightforward by computing the contour integral in (88) for sufficiently small $|q|$ using the residue calculus and (94). □

The infinite sum of Hurwitz zeta values in (88) can be easily converted to a real integral using the definition of Hurwitz zeta or to a contour integral over a Bromwich contour using Mellin summation. The resulting formulas are

$$\begin{aligned}\Gamma(s)\sum_{k=1}^{\infty}\zeta(s,1+2k/\mu) &= \int_0^{\infty}\frac{1}{(\exp(t)-1)}\frac{1}{(\exp(2t/\mu)-1)}\,t^{s-1}\,dt,\\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{2}{\mu}\right)^{-w}\Gamma(w)\Gamma(s-w)\zeta(w)\zeta(s-w)\,dw.\end{aligned} \tag{105}$$

The second integral is convergent provided

$$1<\Re(w)<\Re(s)-1, \tag{106}$$

hence the constant c needs to satisfy $1<c<3/2$ if $\Re(s)=5/2$ as in (88).

In summary, we have given three equivalent representations for the Mellin transform. The formula in (71) is responsible for the small intermittency asymptotic of the Mellin transform. The formula in Theorem 7.1 shows that the Mellin transform is an infinite product generalization of Selberg's finite product. Finally, the third representation given in Theorem 7.2 connects the Mellin transform with the cumulants.

8 Conclusions

We have considered the family of limit log-infinitely divisible (logID) stochastic processes with an emphasis on the limit lognormal process. Our results can be summarized as follows.

All the members of the limit logID family possess three fundamental invariances: those with respect to the decorrelation length, scale, and intermittency parameters. These invariances correspond to, in general, nonlocal equations for the limit process that we refer to as functional Feynman-Kac equations. We have illustrated the technique that converts the former into the latter by explicitly translating the decorrelation length and scale invariances into the corresponding functional equations. The scale invariance equation is a partial integro-differential equation that captures stochastic self-similarity of the limit process, whereas the decorrelation length equation is a nonlocal ordinary integro-differential equation that governs how the limit distribution behaves under a particular change of measure. These equations also determine how the limit distribution depends on the decorrelation length.

The intermittency parameter invariance is of particular importance as the functional equation that it corresponds to is the rule of intermittency differentiation, which is the best known handle on the limit distribution. We have reviewed our results on intermittency differentiation and ensuing intermittency expansions in the limit lognormal case, and then completely determined the structure of the general expansion of the limit distribution. The resulting formal power series is regularized in a way that is consistent with the known way of regularizing the Mellin transform, i.e. the regularized solution corresponds to a valid probability distribution with the correct integral moments. Moreover, the regularized solution is shown to reproduce the general intermittency expansion as its small intermittency asymptotic. It is given in the form of an explicit integral operator.

The structures of either the Mellin transform or operator formulation of the general transform are quite complex. For this reason it is interesting to characterize the limit distribution from an alternative angle. We have succeeded in computing a set of invariants, namely, the cumulants of the logarithm of the limit lognormal distribution, that capture it uniquely. As a byproduct of this calculation, we have given a new representation of the Mellin transform. Its significance is that it splits the Mellin transform into a finite and infinite parts, and represents the infinite part in the form of a single contour integral. The integrand of this integral, which is conceptually the core of the structure of the Mellin transform, involves an infinite sum of Hurwitz zeta values in the integrand.

We have a number of interesting questions that remain unresolved. The rule of intermittency differentiation is not known in the general logID case. In the limit lognormal case, it is unknown how to invert the Mellin transform so as to compute the underlying density or how to compute other transforms explicitly, that is, how to evaluate the action of the aforementioned integral operator on functions other then the exponential. It is not known whether our method of regularization is unique, i.e. whether there is a probability distribution that is different from the one constructed in this paper and having the same integral moments as a function of intermittency and the same asymptotic of the Mellin transform in the limit of small intermittency as those of the limit lognormal distribution. Finally, we find it to be most intriguing that the Riemann zeta function appears in the general intermittency expansion and the Hurwitz zeta function appears in the regularized expression for the Mellin transform, which makes us wonder whether there is some connection between the limit lognormal distribution and analytic number theory.

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